→ Waves steepening (relate kinematic)
→ Eq with discon → RH condition
→ shock structure.

→ Shu
→ L+L

Gasdynamic Shocks

Read: Kulsrud Chapt 6
refs: Landau & Lifshitz, *Fluids*,

Now, proceed from:

— Kinematic waves/shocks, with $V = V(\rho)$ specified ⇒ single equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\underbrace{(\rho V(\rho))}_{Q \to \text{flux}} - \nu \partial_x^2 \rho = 0$$

Key: overtaking, breaking

or

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} - \nu \partial_x^2 \rho = 0$$

Key insight: Characteristics

$$c(\rho) = dQ/d\rho = V(\rho) + \rho V'(\rho)$$

>0 or <0

to:

— gas dynamic shocks.

Simple Waves, etc.

Simple Waves → How describe nonlinear gasdynamic waves, steepening, etc.?

① Consider 1D gasdynamics:

i.e. write cont. mom in $\frac{\partial \alpha}{\partial t} + c(\alpha)\frac{\partial \alpha}{\partial x} = 0$ form.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$p = p(\rho)$

gas adiabatic ⇒ s homogeneous ab-initio (s → specific entropy)

∴ s = const for all times, till shock forms

Now, 1D ; $v = v_x$

$v_y = v_z = 0$ $\qquad c(\rho)$, as in kinematics

so $\qquad \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}\rho v = \dfrac{\partial \rho}{\partial t} + \underbrace{\dfrac{d(\rho v)}{d\rho}}_{c}\dfrac{\partial \rho}{\partial x} = 0$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial v}{\partial t} + \underbrace{\left(v + \frac{1}{\rho}\frac{dp}{dv}\right)}_{c}\frac{\partial v}{\partial x} = 0$$

→ to write in form $\dfrac{d}{dt}\binom{\rho}{v} = 0$ ⇔ exploit characteristics

Now $\qquad \dfrac{\partial \rho / \partial t}{\partial \rho / \partial x} \equiv -\left(\dfrac{\partial x}{\partial t}\right)_\rho$

but, above ⇒

⇒ $\boxed{\left(\dfrac{\partial x}{\partial t}\right)_\rho = \dfrac{d}{d\rho}(\rho v) = v + \rho\dfrac{dv}{d\rho}}$

[from characteristic → ρ equation]

and similarly,

$$\boxed{\left(\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho}\frac{dp}{dv}}$$

Now, since $v = v(\rho)$ [i.e. ρ determines v]

we have: $\frown$, as before

$$\boxed{\left(\frac{\partial x}{\partial t}\right)_\rho = \left(\frac{\partial x}{\partial t}\right)_V}$$

$\Rightarrow$

$$\rho \frac{dV}{d\rho} + V = V + \frac{1}{\rho}\frac{dP}{dV}$$

$$\rho \frac{dV}{d\rho} = \frac{c_s^2}{\rho}\frac{d\rho}{dV} \qquad \underline{dP = c_s^2\, d\rho}$$

$\therefore$ $(dV/d\rho)^2 = c_s^2/\rho^2$

and $dV/d\rho = \pm c_s/\rho$

$$\Rightarrow \boxed{V = \pm\int \frac{c}{\rho}\, d\rho = \pm \int \frac{dP}{\rho c_s}}$$

→ relation between fluid element speed V and density or pressure

→ in accord with expectation, V increases with ρ i.e. for ideal gas

$$P\rho^{-\gamma} = \text{const} \qquad \gamma = 5/3$$

$$dP = c_s^2\, d\rho = \underset{\underset{\text{const.}}{\wr}}{\alpha}\gamma\, \rho^{\gamma-1}\, d\rho$$

$$C_s^2 = \alpha\gamma \rho^{\gamma-1} \qquad C_s = \sqrt{\alpha\gamma}\, \rho^{(\gamma-1)/2}$$

$$V = \pm\int \frac{C_s^2}{\rho C_s} d\rho = \pm\int C_s \frac{d\rho}{\rho}$$

$$= \pm\sqrt{\alpha\gamma}\int \frac{\rho^{1/3}}{\rho} d\rho$$

$$\boxed{V = \pm 3\sqrt{\alpha\gamma}\, \rho^{1/3}}$$

quickie:
$\rho C_s^2 \sim \rho V^2$
$C_s^2 = dP/d\rho$

so

$$\left(\frac{dx}{dt}\right)_V = V + \pm \frac{dP}{\rho\, dV}$$

$$V = \pm\int \frac{dP}{\rho C_s} \Rightarrow dV = \frac{dP}{\rho C_s}$$

$$dP/dV = \rho C_s$$

$$\Rightarrow \boxed{\left(\frac{dx}{dt}\right)_V = V \pm C_s(V)}$$

since $\rho = \rho(V)$

$$\boxed{x = t\left[V \pm C_s(V)\right] + f(V)}$$

thus, have "simple wave" solution:

$$\boxed{x = t\left[V \pm c_s(V)\right] + f(V)}$$

{ no scale in problem

obviously, for linearized limit;

$V \ll c_s$.

$$x = t\left[\cancel{V} \pm c_s\right] + \cancel{f(V)} + x_0$$

$$x = x_0 \pm c_s t$$

sign → direction

⇝ Alternative approach → scale independence ⇓

Why "simple"? → 1 D Similarity Flow
(self-similar)
→ characteristic velocity
→ $\boxed{\underline{\underline{\text{no}}}\ \text{characteristic length}}$

scale-independent.

so... assume all quantities depend only on $\varepsilon = x/t$

$\varepsilon = x/t$

$$\Rightarrow \quad \frac{\partial}{\partial x} = \frac{1}{t}\frac{d}{d\varepsilon}$$

i.e. $\begin{pmatrix}\rho \\ V\end{pmatrix} = f(\varepsilon)$

$\varepsilon = (x/t)$

$$\frac{\partial}{\partial t} = -\frac{\varepsilon}{t}\frac{d}{d\varepsilon}$$

so

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial V}{\partial x} + V\frac{\partial \rho}{\partial x} = 0$$

{ n.b. $\frac{x}{t}$ is "velocity" formed by x, t in absence scales.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{-1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \begin{cases} -\frac{\varepsilon}{t}\rho' + \frac{\rho}{t}v' + \frac{v}{t}\rho' = 0 \\ \\ -\frac{\varepsilon}{t}v' + \frac{v}{t}v' = -\frac{c_s^3}{t}\frac{\rho'}{\rho} \end{cases} \qquad ' = \frac{d}{d\varepsilon}$$

const entropy

$$\Rightarrow \begin{cases} (v-\varepsilon)\rho' + \rho v' = 0 \\ (v-\varepsilon)v' = -c_s^2\frac{\rho'}{\rho} \end{cases}$$

$\therefore$ as $\omega(k)$, treat ε as an eigenvalue, tbd:

$$\begin{cases} (v-\varepsilon)\rho' + \rho v' = 0 \\ -\frac{c_s^2}{\rho}\rho' + (v-\varepsilon)v' = 0 \end{cases}$$

$$(v-\varepsilon)^2 = c_s^2$$

$\Rightarrow$

$$\varepsilon = v \mp c_s = \frac{x}{t}$$

$\Rightarrow \quad x = (v \pm c_s)\, t$ ✓

From "eigenvector", relate v, ρ etc.

i.e. $\quad v - \varepsilon = -c_s$

and $\quad (v-\varepsilon)\rho' + \rho v' = 0$

$\quad -c_s \rho' + \rho v'$

$\Rightarrow \quad \rho\, dv = c_s\, d\rho$

$\therefore \quad \boxed{dv/d\rho = c_s/\rho}$

$\Rightarrow \quad \boxed{v = \int c_s\, d\rho/\rho = \int dp/c_s\rho}$

- equivalent to previous result with $f(v) = 0$ (Restricted version of "simple wave")
- can also write as

$$\boxed{v = \int \sqrt{-dp\, dV}}$$

$d(1/\rho) = dV = -\frac{1}{\rho^2}\, d\rho$

$dp = c_s^2\, d\rho$

$$v = \int \left(c_s^2\, d\rho\, \frac{d\rho}{\rho^2} \right)^{1/2}$$

$$= \int \frac{d\rho}{\rho}\, c_s$$ ✓

→ Physics of Simple Wave (in Gasdynamics)

→ how, when break

→ shock formation

$$X = \cancel{t}\left[V \pm c_s(V)\right] + f(V)$$

- similarity flow is simple wave with $f(V) = 0$

- can write general solution for simple wave, for adiabatic process

i.e. $P\rho^{-\gamma} = \text{const}$

$T\rho^{-(\gamma-1)} = \text{const}$ $\qquad T = c_s^2 = \rho^{\gamma-1}$

$\Rightarrow \rho T^{1/\gamma-1} = \text{const.}$ $\qquad \boxed{P = \rho T}$

but $c_s^2 \sim T$ $\quad (c_s \sim \sqrt{T})$ $\Rightarrow$

$$\boxed{\boxed{\rho = \rho_0 (c/c_0)^{2/\gamma-1}}}$$

$dP/d\rho = c_s^2 = T$

but $\boxed{V = \pm\int c \frac{d\rho}{\rho}}$

$$d\rho = \rho_0 \frac{2}{\gamma-1}\frac{dc}{c_0}\left(\frac{c}{c_0}\right)^{\left(\frac{2}{\gamma-1}-1\right)}$$

$\Rightarrow$

$$= \frac{\rho_0 2}{\gamma-1}\frac{dc}{c_0}\left(\frac{c}{c_0}\right)^{\left(\frac{2}{\gamma-1}-1\right)}$$

$$\left(\frac{2}{\gamma-1}\right) - \left(\frac{\gamma-1}{\gamma-1}\right) = \frac{-(\gamma-3)}{\gamma-1}$$

$$V = \pm \int \frac{c \,\frac{2}{\gamma-1}\, \rho_0 \frac{dc}{c_0} \left(\frac{c}{c_0}\right)^{(2/\gamma-1 \,-1)}}{\rho_0 \,(c/c_0)^{2/\gamma-1}}$$

$$= \pm\frac{2}{\gamma-1}\int dc \quad = \pm \frac{2}{\gamma-1}(c-c_0)$$

$$\Rightarrow \quad \boxed{V = \pm \frac{2}{\gamma-1}(c-c_0)}$$

i.e.

$$\boxed{\begin{aligned} c &= c_0 \pm \tfrac{1}{2}(\gamma-1)\, V \\ \rho &= \rho_0\left(1 \pm \tfrac{1}{2}(\gamma-1)\, V/c_0\right)^{2/\gamma-1} \\ p &= p_0\left(1 \pm \tfrac{1}{2}(\gamma-1)\, V/c_0\right)^{2\gamma/\gamma-1} \end{aligned}}$$

Show

then can reduce simple wave expression:

$$x = t\left[V \pm c_0(V)\right] + f(V)$$

$$\Rightarrow \quad x = t\left[V \pm \left(c_0 \pm \tfrac{1}{2}(\gamma-1)V\right)\right] + f(V)$$

+ sign for compression (steepening):

$$\boxed{x = t\left[\pm c_0 + \tfrac{1}{2}(\gamma+1)v\right] + f(v)}$$

⇒ point on wave profile moves at speed
$u = v \pm c_s$

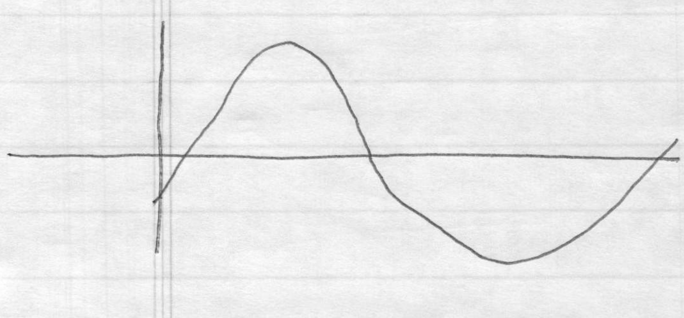
→ have shown $du/d\rho > 0$
i.e. speed increases with density

so

⇒ if wave in $+x$, then $d\rho/dx < 0$ anywhere in i.c. ⇒

⇒ discontinuity will form ⇒ shock!

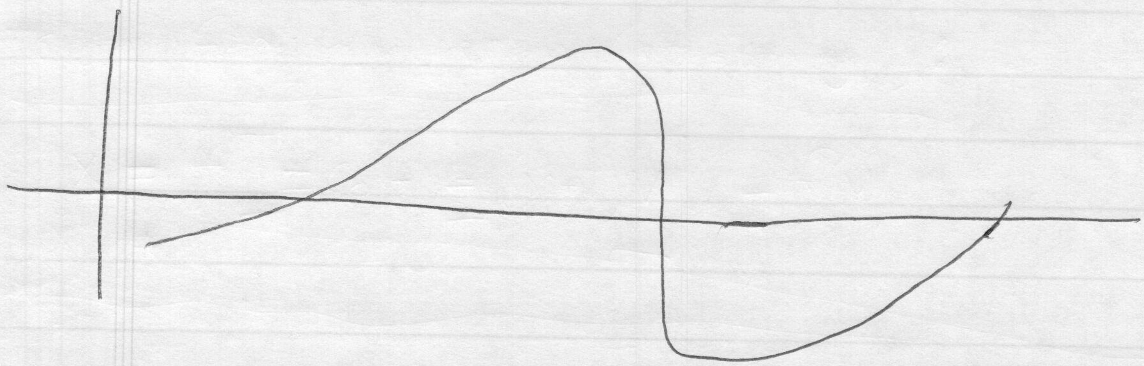
i.e. via over-taking and breaking mechanism



wave

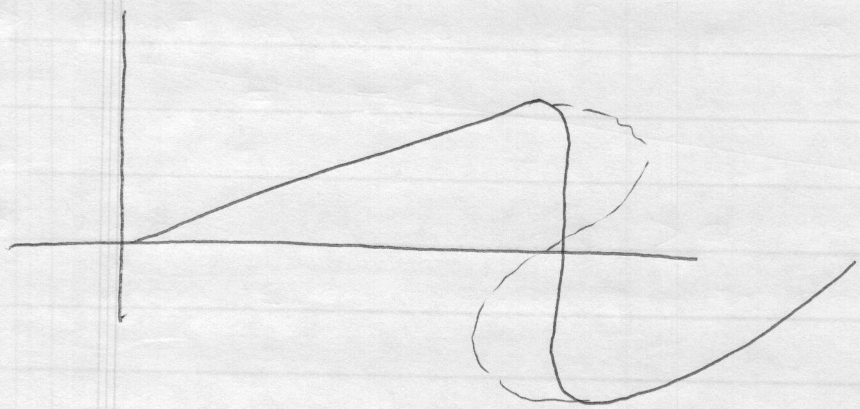
as before

⇒



steepening

→



shock

(dissipation resolves singularity)

→ when will breaking occur?

⇒ multi-valued position, i.e. a la' characteristic crossing

$$\left(\partial x/\partial v\right)_t = 0, \quad \left(\partial^2 x/\partial v^2\right)_t = 0$$ inflection

$$x = t\left[\pm c_0 + \frac{1}{2}(\gamma+1)v\right] + f(v)$$

$$\left.\frac{\partial x}{\partial v}\right|_t = \frac{1}{2}(\gamma+1)t + f'(v) = 0$$

$$\boxed{t_{\substack{break\\shock}} = -2f'(v)/(\gamma+1)}$$

critical values

(need $f(v)' < 0$ i.c.)

and $f''(v) = 0$

now → have discussed how shocks form → now convergence

② Shocks – Flows with Discontinuity

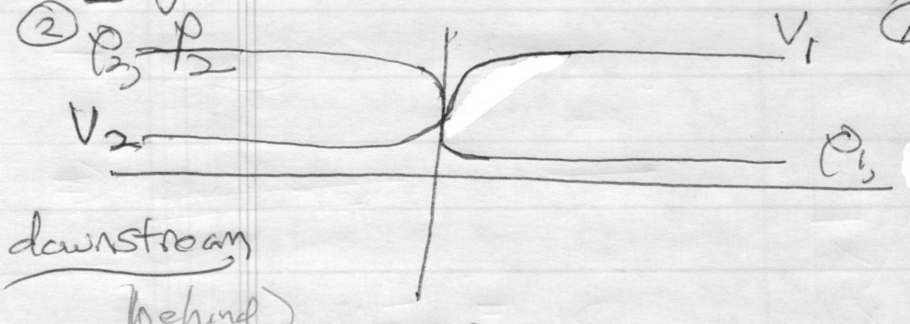
- once waves steepen and break/shock
  ⇒ have flow with discontinuities

  distinguishing feature of shocks. → "the point"

- "shock" – { localized region of rapid change / discontinuity

flow ⟵

i.e. generic



upstream (ahead)

downstream (behind)

{ converter idea

$x = 0$

↳ location of shock in its rest frame (stationary)

layer thickness ≈ l_{mfp}

Now, have conservation equations for ideal fluid:

i.e. recall

$$\underset{\text{shock speed}}{u} = \frac{\overbrace{Q(\rho_1) - Q(\rho_2)}^{\text{flux discon.}}}{(\rho_1 - \rho_2)}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{V}) = 0 \quad \rightarrow \text{continuity}}$$

specific enthalpy (incl PV work)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \rho\epsilon\right) + \nabla\cdot\left(\rho \underline{V}\left(\frac{1}{2}V^2 + \frac{\gamma P}{(\gamma-1)\rho}\right)\right) = 0$$

→ energy

$$\frac{\gamma}{\gamma-1}P = \frac{P}{\gamma-1} + P$$

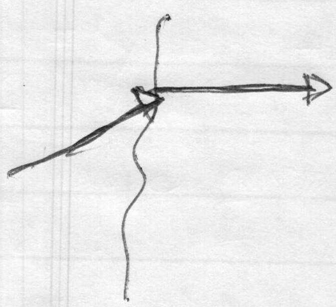
energy density

PV work on surroundings

stress tensor

$$\frac{\partial}{\partial t}\rho V_i = -\frac{\partial}{\partial x_k}\Pi_{ik}$$

$$\Pi_{ik} = P\delta_{ik} + \rho V_i V_k$$



x=0

General

so $V^{(2)}_{y,z} = V^{(1)}_{y,z}$

tangential components continuous

so

→ Integrating conservation equations

→ work in shock frame ⇒ $U = 0$

continuity ⇒ $$\boxed{\rho V_x \Big|_{2} = \rho V_x \Big|_{1}}$$

energy conservation ⇒

$$\rho V_n \left(\frac{1}{2} V^2 + \frac{\gamma P/\rho}{\gamma - 1}\right)_2 = \rho V_n \left(\frac{V^2}{2} + \frac{\gamma P/\rho}{\gamma - 1}\right)_1$$

but $\rho V_n \Big|_2 = \rho V_n \Big|_1$

$V_{y,z\,2}^2 = V_{y,z\,1}^2$ tangential continuity.

⇒ $$\boxed{\left(\frac{V_x^2}{2} + \frac{\gamma P/\rho}{\gamma - 1}\right)_2 = \left(\frac{V_x^2}{2} + \frac{\gamma P/\rho}{\gamma - 1}\right)_1}$$

and momentum conservation ⇒

$$\boxed{\left(p + \rho V_x^2\right)\Big|_{2} = \left(p + \rho V_x^2\right)\Big|_{1}}$$

(again, normal component varies, only).

$\Rightarrow$ have **3 Rankine-Hugoniot jump/continuity conditions:** $\rightarrow$ critical balance relations

$$[\ \ ] = (\ )_2 - (\ )_1$$

$$\boxed{\begin{aligned} &[\rho V_x] = 0 \\ &\left[\frac{V_x^2}{2} + \frac{\gamma p/\rho}{\gamma - 1}\right] = 0 \\ &[\rho V_x^2 + p] = 0 \end{aligned}}$$

in shock frame

if in fixed coordinate,

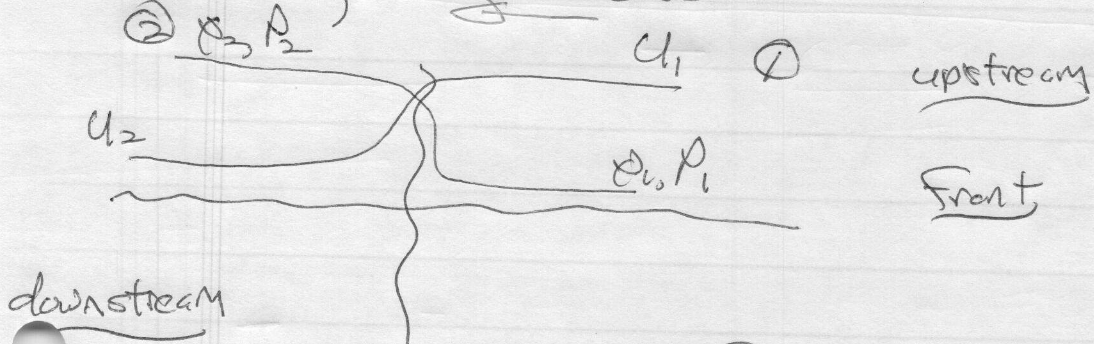
$$\boxed{V_x \rightarrow \underbrace{V_n}_{\text{normal } V \text{ in fixed coords}} - \underbrace{U}_{\text{shock velocity}}}$$

R-H Conditions ⇒

(3) Shock Structure → shock as upstream → downstream transition

→ Can derive very general relation between thermodynamic quantities on 2 sides of shock discontinuity, using R-H conditions.

→ as before, consider case:



back   discontinuity

$$\begin{cases} V_{tan} = 0 \\ u = 0, \text{ in frame} \end{cases}$$

then Rankine - Hugoniot conditions ⇒

$$\left.\begin{aligned} \rho_1 V_1 &= \rho_2 V_2 = j \\ P_1 + \rho_1 V_1^2 &= P_2 + \rho_2 V_2^2 \\ w_1 + \frac{V_1^2}{2} &= w_2 + \frac{V_2^2}{2} \end{aligned}\right]$$

$j \equiv$ mass flux density at surface of discontinuity

where $w = \dfrac{\gamma P}{(\gamma-1)\rho} \equiv$ enthalpy density

Now, $\bar{V}_1 = 1/\rho_1$, $\bar{V}_2 = 1/\rho_2$ } specific volumes

$$\Rightarrow \boxed{\begin{array}{l} \underset{\text{velocity}}{V_1} = j\,\bar{V}_1 \\ V_2 = j\,\bar{V}_2 \end{array}}$$

→ relates flow speeds to flux and volumes

so

$$P_1 + j^2\bar{V}_1 = P_2 + j^2\bar{V}_2$$

and

$$\boxed{j^2 = (P_2 - P_1)\,/\,(\bar{V}_1 - \bar{V}_2)}$$

→ relates flux and speed to pressure difference and volumes difference on both sides / across shock

→ as $j^2 > 0$ must either have:

$$\begin{pmatrix} P_2 > P_1 \\ V_1 > V_2 \end{pmatrix} \quad \underline{\text{or}} \quad \begin{pmatrix} P_2 < P_1 \\ V_1 < V_2 \end{pmatrix}$$

$(\rho_2 > \rho_1)$ $\qquad$ $(\rho_2 < \rho_1)$

will see that only $\boxed{\begin{array}{l} P_2 > P_1 \\ \bar{V}_1 > \bar{V}_2 \end{array}}$ is physical.

Why? $\Rightarrow$ $\boxed{\text{why only } \begin{matrix} P_2 > P_1 \\ \rho_2 > \rho_1 \end{matrix} \text{ physical?}}$

** Answer: $\Rightarrow$ $\boxed{\text{Shock must increase entropy}}$ **

i.e. $S_2 > S_1$

$\Rightarrow$ as
- (1) entropy can only increase during gases motion
- (2) microscopic diffusion (ν, K, etc.) in shock effect entropy increase on scale of shock thickness → heating
- (3) ammount of entropy increase set by macro jump conditions

$\Rightarrow$ and, $P_2 > P_1$ $\qquad P \sim \rho T$

$V_1 > V_2$ $(\rho_2 > \rho_1)$

clearly correspond to $\left\{ \begin{matrix} \text{compression} \\ \text{heating} \end{matrix} \right.$

$\Rightarrow$ entropy increase.

[n.b. for rigorous proof see: Landau Section 87)

$\therefore$ $\boxed{S_2 > S_1 \Rightarrow \begin{matrix} P_2 > P_1 \\ \rho_2 > \rho_1 \end{matrix}}$ is only physical solution

n.b. need: $\boxed{\begin{matrix} V_1/C_{S1} > 1 \\ V_2/C_{S2} < 1 \end{matrix}}$, too

Now, can go further, i.e.

$$v_1 - v_2 = j(V_1 - V_2)$$

$$j^2 = (p_2 - p_1)/V_1 - V_2$$

$$\Rightarrow \boxed{\underbrace{v_1 - v_2}_{\text{velocity difference}} = \sqrt{(p_2 - p_1)(V_1 - V_2)}}$$

velocity difference → jump in velocity across shock

$p_2 > p_1 \Rightarrow$
$v_2 < v_1$
$\Leftrightarrow$ + sqr. root

and similarly,

$$w_1 + \frac{v_1^2}{2} = w_2 + \frac{v_2^2}{2}$$

$$\Rightarrow w_1 + \frac{j^2}{2} V_1^2 = w_2 + \frac{j^2}{2} V_2^2$$

and since $j^2 = (p_2 - p_1)/(V_1 - V_2)$

$$\Rightarrow \boxed{w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(p_2 - p_1) = 0}$$

~~enthalpy~~ enthalpy jump.

Now, let $\overset{\text{enthalpy}}{W} = \underset{\text{internal energy}}{\epsilon} + p\bar{V}$

$$\frac{p/\rho}{(\gamma-1)} + \frac{p}{\rho} = \frac{\gamma}{\gamma-1} p/\rho$$

$\Rightarrow$

$$\boxed{\epsilon_1 - \epsilon_2 + \frac{1}{2}(\bar{V}_1 - \bar{V}_2)(p_1 + p_2) = 0}$$

What have we gained from this?

given inflow state thermo variables $\underline{p_1}$, $\underline{\bar{V}_1}$

then

$$w_1 - w_2 + \frac{1}{2}(\bar{V}_1 + \bar{V}_2)(p_2 - p_1) = 0$$

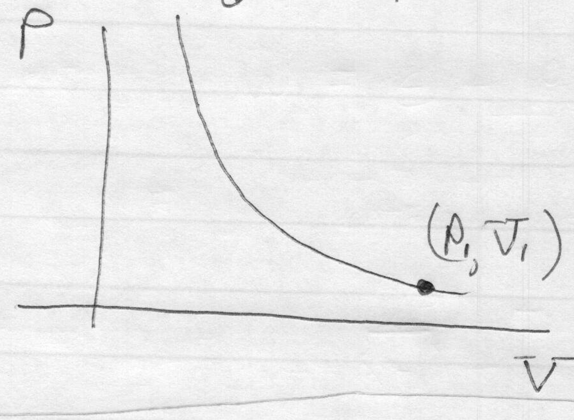
and/or

$$\epsilon_1 - \epsilon_2 + \frac{1}{2}(\bar{V}_1 - \bar{V}_2)(p_2 - p_1) = 0$$

$\Rightarrow$ outflow/downstream state relation between $\underline{p_2, \bar{V}_2}$.

curve of possible $p_2, \bar{V}_2$ points

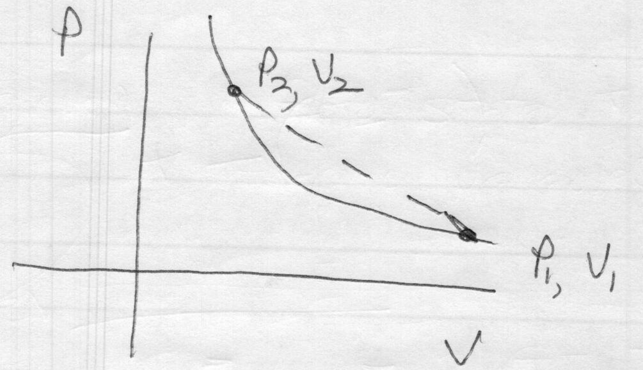
$\Rightarrow$



$\leadsto$ { shock adiabatic / Hugoniot adiabatic }

- Hugoniot adiabatic is curve on which downstream thermodynamic states (p_2, V_2 fall), for given p_1, V_1

- can graphically determine { flux j / velocity }



$$\frac{p_2 - p_1}{\bar{V}_2 - \bar{V}_1} = \text{slope} = -j^2$$

i.e. Hugoniot adiabat + flux $\Rightarrow$ { final state / velocity }

$\therefore$ { flux j / velocity v } determined at each point of shock adiabatic

[Now, can use shock adiabatic relations/equations to characterize discontinuity.

→ Characterizing the Discontinuity ⟺ $\left\{ \begin{array}{l} \underline{\text{Jump Ratios}} \\ \underline{\text{for}} \\ \text{Polytropic Gas} \\ \left[p\rho^{-\gamma} = \text{const} \right] \end{array} \right.$

Now, have shown:

$$w_1 - w_2 + \frac{1}{2}(V_1 + V_2)(p_2 - p_1) = 0$$

$$w = \frac{\gamma p V}{\gamma - 1} = \frac{\gamma\, p/\rho}{\gamma - 1} = \frac{c_s^2}{\gamma - 1}$$

⇒

$$\frac{\gamma}{\gamma-1}\left[p_1 V_1 - p_2 V_2\right] + \frac{1}{2}(V_1 + V_2)(p_2 - p_1) = 0$$

$$\frac{\gamma}{\gamma-1}\left[p_1 - p_2 \frac{V_2}{V_1}\right] + \frac{1}{2}\left(1 + \frac{V_2}{V_1}\right)(p_2 - p_1) = 0$$

$$\frac{\gamma}{\gamma-1} p_1 + \frac{1}{2}(p_2 - p_1) = \frac{\gamma}{\gamma-1} p_2 \frac{V_2}{V_1} - \frac{V_2}{V_1}\frac{(p_2 - p_1)}{2}$$

$$\frac{\gamma}{\gamma-1} p_1 + \frac{1}{2}(p_2 - p_1) = \frac{V_2}{V_1}\left[\frac{\gamma}{\gamma-1} p_2 - \frac{p_2 + p_1}{2}\right]$$

⇒

$$\boxed{\frac{V_2}{V_1} = \frac{2\gamma p_1 + (\gamma-1)(p_2 - p_1)}{2\gamma p_2 - (\gamma-1)(p_2 - p_1)}}$$

$\frac{\rho_2}{\rho_1}\Big|_{\text{max}} = \frac{\gamma+1}{\gamma-1} = 4$ follows

(#) so finally,

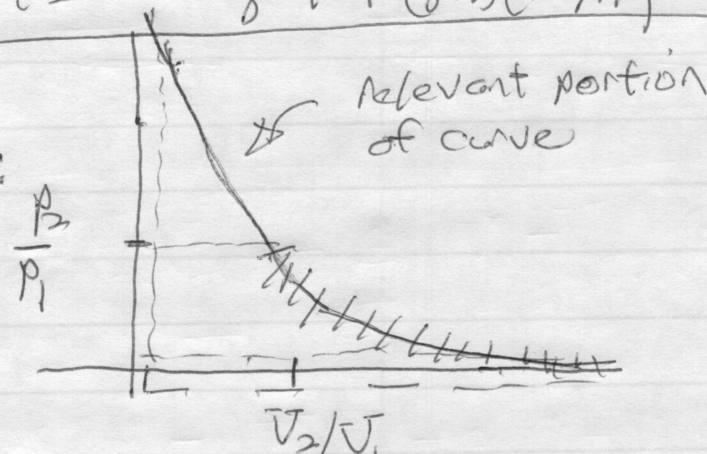
$$\boxed{\frac{V_2}{V_1} = \frac{(\gamma+1)\,p_1 + (\gamma-1)\,p_2}{(\gamma-1)\,p_1 + (\gamma+1)\,p_2}}$$

Volume/density ratio ⇒ $\boxed{\text{compression ratio}}$

$$\Rightarrow \boxed{\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma+1 + (\gamma-1)(p_2/p_1)}{\gamma-1 + (\gamma+1)(p_2/p_1)}}$$

[note: $p_2 \gg p_1$ $\Rightarrow \frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} = 4$

Graphically:



hyperbola

$p_2 > p_1$

$\rho_2 > \rho_1$

x-asymptote: $p_2/p_1 \to \infty \Rightarrow V_2/V_1 = (\gamma-1)/(\gamma+1)$

y-asymptote: $V_2/V_1 \to \infty \Rightarrow p_2 = -(\gamma-1)/(\gamma+1)$

then can also extract temperature, velocity, flux etc.

i.e. (e) $p = \rho T \Rightarrow T = pV$

$$\boxed{\frac{T_2}{T_1} = \frac{p_2}{p_1}\left(\frac{V_2}{V_1}\right) = \frac{p_2}{p_1}\left[\frac{(\gamma+1)\,p_1 + (\gamma-1)\,p_2}{(\gamma-1)\,p_1 + (\gamma+1)\,p_2}\right]}$$

→ Temperature ratio: note no limit for $p_2 \gg p_1$

For flux j $\quad j^2 = (p_2 - p_1)/(\mathcal{V}_1 - \mathcal{V}_2)$ *

(Hugoniot)

and using $\dfrac{\mathcal{V}_2}{\mathcal{V}_1} = \dfrac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}$

$$\Rightarrow \boxed{j^2 = \left[(\gamma-1)p_1 + (\gamma+1)p_2\right]/2\mathcal{V}_1}$$

and,

$$v_1^2 = j^2 \mathcal{V}_1^2 \quad , \quad v_2^2 = j^2 \mathcal{V}_2^2$$

so using expressions for: $-j^2$
$-\mathcal{V}_2/\mathcal{V}_1$

$\Rightarrow$

$$\boxed{\begin{aligned} v_1^2 &= \frac{1}{2}\mathcal{V}_1\left\{(\gamma-1)p_1 + (\gamma+1)p_2\right\} \\ &= \frac{1}{2}\left(c_{s1}^2/\gamma\right)\left[\gamma-1 + (\gamma+1)p_2/p_1\right] \end{aligned}}$$

$$\boxed{\begin{aligned} v_2^2 &= \frac{1}{2}\mathcal{V}_1\left\{(\gamma+1)p_1 + (\gamma-1)p_2\right\}^2 / \left\{(\gamma-1)p_1 + (\gamma+1)p_2\right\} \\ &= \frac{1}{2}\frac{c_2^2}{\gamma}\left[\gamma-1 + (\gamma+1)p_1/p_2\right] \end{aligned}}$$

Now, often convenient to describe density ratio, etc. in terms of Mach number of upstream flow

$$M_1 = V_1 / Cs_1$$

now $\quad V_1^2/Cs_1^2 = M_1^2 = \frac{1}{2\gamma}\left[(\gamma-1)+(\gamma+1)P_2/P_1\right]$

so can show

$$\boxed{\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{V_1}{V_2} = (\gamma+1)\, M_1^2 \Big/ \left[(\gamma-1)M_1^2 + 2\right] \\ \frac{P_2}{P_1} &= \left(2\gamma M_1^2 \big/ \gamma+1\right) - \frac{(\gamma-1)}{(\gamma+1)} \\ \frac{T_2}{T_1} &= \left\{2\gamma M_1^2 - (\gamma-1)\right\}\left\{(\gamma-1)\, M_1^2 + 2\right\} \Big/ (\gamma+1)^2 M_1^2 \end{aligned}}$$

and sometimes useful to use:

$$M_2^2 = \left\{2 + (\gamma-1)\, M_1^2\right\} \Big/ \left\{2\gamma M_1^2 - (\gamma-1)\right\}$$

Interesting to note case of strong shock waves
$\Rightarrow$

$$M_1 \gg 1$$

i.e.

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}} \rightarrow \sim 4, \text{ maximum}$$

(limited by eqn. of state)

$$\frac{P_2}{P_1} = 2\gamma M_1^2/\gamma+1 \rightarrow \sim M_1^2$$

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2 \rightarrow \sim M_1^2$$

heating, pressure $\sim M_1^2$

also check: $M_1 = 1$

$$\rho_2/\rho_1 = 1 \checkmark, \quad P_2/P_1 = 1 \checkmark, \quad T_2/T_1 = 1 \checkmark$$

So

① $\rightarrow$ simple waves $\Rightarrow$ shock formation

② $\rightarrow$ jump conditions, shock adiabati $\Rightarrow$ shock {physics, structure}

③ $\rightarrow$ shock jump conditions $\Rightarrow$ shock properties.

So ...
shock as heater...

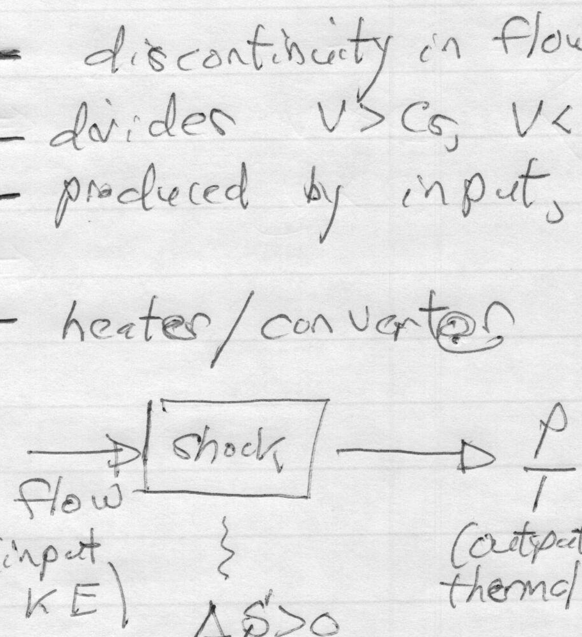
MHD Shocks

Read:
- B+S, Chapt. 5
- L+L, ECM Sect. 69-73

a.) Review of Gasdynamic shocks

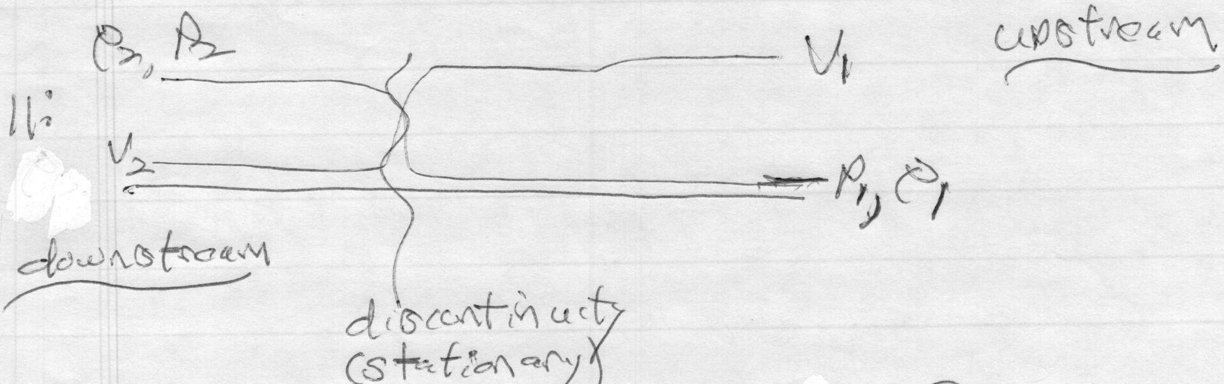
shock: - discontinuity in flow (not unique sort)
- divides $V > c_s$, $V < c_s$ regions
- produced by input, i.c.'s i.e. {explosion, wave steepening

(*) - heater / converter



{ a/a! house heated by stream K.E.

recall:



typical shock parameters: $\begin{cases} r = \rho_2/\rho_1 \to \text{compression ratio} \\ R = P_2/P_1 \to \text{strength} \\ \text{etc.} \end{cases}$

i.e. $\left.\begin{array}{l} r = r(M_1) \\ R = R(M_1) \end{array}\right\}$ "the answer"

- exact NL solutions! → tractability from piecewise continuity!

rules of operation: $\quad s_2 > s_1$

R-H conditions
$$\begin{cases} [\rho V_n] = 0 \\ \left[w + \frac{V_n^2}{2}\right] = 0 \\ \left[p + \rho \frac{V_n^2}{2}\right] = 0 \end{cases} \qquad w = \gamma p / \rho(\gamma - 1)$$

Performance:

$M = V_1 / C_{s_1} >> 1$

(upstream Mach)

$$\begin{cases} r = r(M) \rightarrow \frac{\gamma+1}{\gamma-1} \\ R = R(M) \rightarrow 2\gamma M^2/(\gamma+1) \\ T_2/T_1 \rightarrow \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M^2 \end{cases}$$

indep M (constr. by flux, mom-flux balance)

n.b. for large M #:

$$w_1 + \frac{V_1^2}{2} = w_2 + \frac{V_2^2}{2} \qquad \text{(energy flux balance)}$$

$$\Rightarrow \frac{V_1^2}{2} \sim w_2 \qquad \Rightarrow M^2 C_{s_1}^2 \sim C_{s_2}^2$$

(all KE) (all ThE) $\qquad T_2/T_1 \sim M^2$

aside: Is shock only form of discontinuity?

need Π continuous accross discontinuity

$$\Pi_i = p n_i + \rho V_i V_k n_k$$

i.e. $\underline{\underline{\Pi}} = p\,\underline{\underline{I}} + \rho\,\underline{V}\,\underline{V}$

$\underline{\underline{\Pi}} \cdot \underset{\S}{\hat{n}} \equiv$ Momentum flux thru surface

unit normal
to shock surface

$\Pi_x = p + \rho V_x^2 \qquad \Pi_y = \rho V_x V_y$

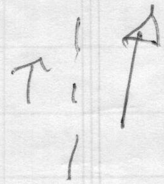
so $[\rho V_x V_y] = 0 \quad \Rightarrow$ either:

$[\rho V_x] \neq 0 \Rightarrow [V_y] = 0$

(tangential flow continuous at shock)

or $[\rho V_x] = 0 \Rightarrow [V_y] \neq 0$

tangential discontinuity



i.e. $\Rightarrow$ KH, eddy, vortex etc.

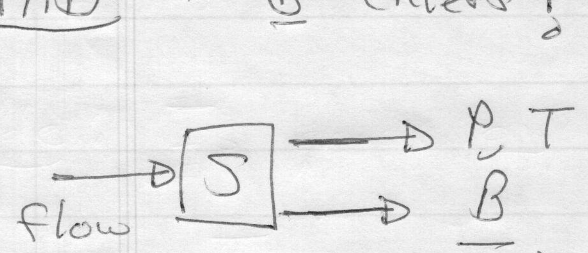
n.b. shock $\Leftrightarrow$ acoustic wave

t-discon $\Leftrightarrow$ vortex mode

(n.b. Bomb $\rightarrow$ γ's ionize ahead of blast
$\rightarrow$ shock in plasma)

G-D shock $\Rightarrow$ "evolved sound wave"

MHD - $\underline{B}$ enters!



(KE) (Th E)
(Magn E)

(new channel!)

$\Rightarrow$ so MHD shock...

MHD shock $\Rightarrow$ evolved $\begin{cases} \text{fast} \\ \text{intermediate} - ? \\ \text{slow} \end{cases}$

as before, will elucidate via extremes:

fast $\rightarrow$ $\underline{V}_1 \perp \underline{B}$ $\rightarrow$ perpendicular shock
(from magnetosonic wave)

slow $\rightarrow$ $\underline{V}_1 \parallel \underline{B}$ $\rightarrow$ parallel shock
(from $\parallel$ acoustic wave)

then consider $\rightarrow$ oblique shock

and intermediate wave $-$ $\begin{cases} \text{transverse} \\ \text{EM} \end{cases}$

$\rightarrow$ rotational discontinuity

First: Jump Conditions for MHD shock ($\hat{n} = \hat{x}$)

(i)

$$\underline{\nabla} \cdot \underline{B} = 0 \quad \Rightarrow \quad [B_x] = 0$$

(ii)

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \quad \Rightarrow \quad [E_y] = 0 \qquad \underline{E} = -\underline{v} \times \underline{B}/c$$

$$[E_z] = 0$$

$\therefore$

$$[V_z B_x - V_x B_z] = 0$$

$$[v_x B_y - v_y B_x] = 0$$

$\left(\frac{d}{dx} E_{y,z} = 0\right)$

$\gamma P/(\gamma-1)\rho$

(c.c.) $\left[\rho v_x\right] = 0$

(e.v) $\dfrac{\partial}{\partial t}\left(\underbrace{\epsilon}_{\to P/(\gamma-1)} + \frac{1}{2}\rho v^2 + \frac{B^2}{8\pi}\right) + \nabla\cdot\left(\rho \underline{v}\left[\frac{v^2}{2} + \overbrace{w}^{\gamma P/(\gamma-1)\rho}\right] + \frac{c}{4\pi}(\underline{E}\times\underline{B})\right) = 0$

(from energetics)

so

$$\left[\rho v_x\left[\frac{v^2}{2} + w\right] + \frac{c}{4\pi}(\overbrace{\underline{E}}^{-\frac{\underline{v}\times\underline{B}}{c}}\times\underline{B})_x\right] = 0$$

↳ Poynting flux

(v) $\dfrac{\partial}{\partial t}\rho v_i = -\underline{\nabla}\cdot T_{ik}$

↳ stress tensor (full)

Reyn. tension total pressure.

$$T_{ik} = \left(\rho\underline{v}\,\underline{v} - \frac{\underline{B}\,\underline{B}}{4\pi} + \underline{\underline{I}}\left(P + \frac{B^2}{8\pi}\right)\right)_{i,k}.$$

$(\Pi_i = T_{i,k}\, n_k)$

so (x,x) component:

$$\left[\rho v_x^2 - \frac{B_x B_x}{4\pi} + P + \frac{B^2}{8\pi}\right] = 0$$

$$\therefore \quad \left[\rho v_x^2 + \underbrace{\frac{B_y^2 + B_z^2}{8\pi}}_{B_t^2/8\pi} + P\right] = 0$$

but now off-diagonal terms non-trivial, i.e.

$$\Pi_y = \rho v_x v_y - \frac{B_x B_y}{4\pi}$$

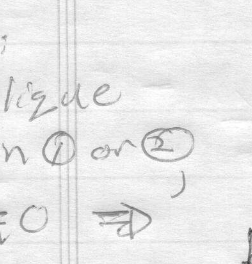
so $[\Pi_y] = 0$ $\qquad \left[\rho v_x v_y - \frac{B_x B_y}{4\pi}\right] = 0$

$[\Pi_z] = 0$ $\qquad \left[\rho v_x v_z - \frac{B_x B_z}{4\pi}\right] = 0$

Note: – no longer have $[\rho v_x v_y] = 0$

– point is that magnetic stresses deliver impulse to fluid element as it crosses the shock.....

i.e. now $[v_t] \neq 0$, due $\underline{J} \times \underline{B}$



$$\Delta(\rho v_y) \sim F_{J\times B}\, \tau_c$$

τ_c → time fluid element to cross shock

$$\sim B_x J_z \tau_c$$

$\tau_{crossing} \sim \delta / v_x$

$$\sim B_x J_z \frac{\delta}{v_x}$$

δ → thickness, v_x → flow

point: if oblique field on ① or ②, $B_{y_{1,2}} \neq 0 \Rightarrow$ tangential impulse imparted.

but $J_z \sim B_y/\delta$

$$\Delta(\rho v_y) \sim \frac{B_x B_y}{v_x} \Rightarrow \left[\rho v_x v_y - \frac{B_x B_y}{4\pi}\right] = 0$$

so, writing out full conditions:

($\hat{n}$ = normal to plane of interface)

$$[\rho V_n] = 0$$

$$\left[\rho \underline{V} V_n + \left(p + \frac{B^2}{8\pi}\right)\hat{n} - \frac{(\underline{B}\cdot\hat{n})}{4\pi}\underline{B}\right] = 0$$

↳ magn. tension

$$\left[\rho V_n\left(\frac{V^2}{2} + w\right) + \hat{n}\cdot\frac{c}{4\pi}(\underline{E}\times\underline{B})\right] = 0$$

or

$$\left[\rho V_n\left(\frac{V^2}{2} + w + \frac{B^2}{4\pi}\right) - \frac{(\underline{B}\cdot\hat{n})(\underline{V}\cdot\underline{B})}{4\pi}\right] = 0$$

#!

$$[B_n] = 0$$

$$[\hat{n}\times\underline{E}] = [\hat{n}\times\underline{V}\times\underline{B}] = 0.$$

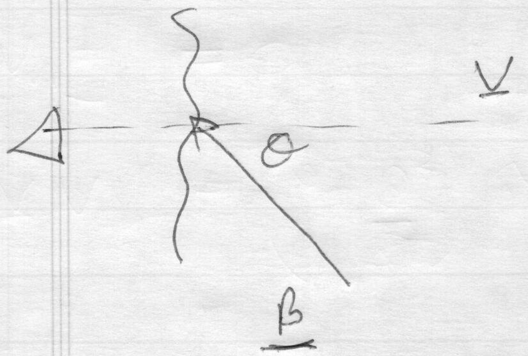
⇒ full set of MHD jump conditions.

Now consider:

a) parallel field, flow

b) perpendicular field, flow.

in general, have:



$\theta = 0^\circ \Rightarrow$ parallel shock

$\underline{V} \parallel \underline{B}$

$\theta = \pi/2 \Rightarrow \perp$ shock

(a) $\theta = 0^\circ$ $\quad \underline{B} = B_x \hat{x}$ (only) $\quad$ (both sides)

(parallel shock)

$[B_x] = 0$, $[\rho V_x] = 0$,

$$\left[\rho V_x^2 + P + \cancel{\frac{B_y^2 + B_z^2}{8\pi}}\right] = 0$$

$$[\hat{n} \times \underline{V} \times \underline{B}] = 0$$

$$\left[\rho V_x\left(\frac{V^2}{2} + w + \cancel{\frac{B^2}{4\pi}}\right) - \cancel{\frac{B_x V_x B_x}{4\pi}}\right] = 0$$

then can simplify, as $\underline{B}$ field drops out, i.e.

$$[\rho V_x] = 0$$

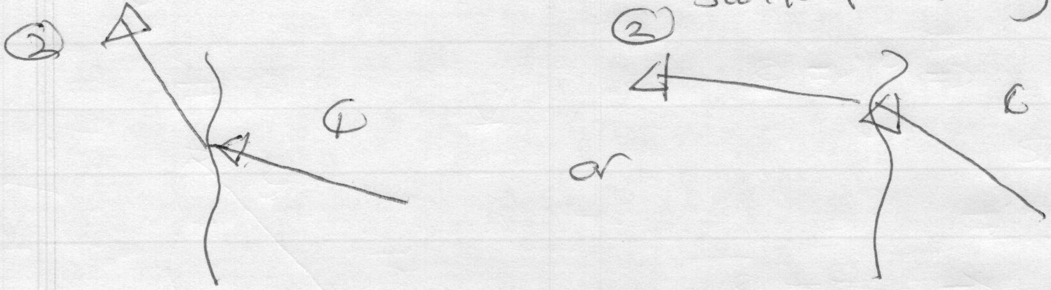
$$[\rho V_x^2 + P] = 0$$

$$\left[\frac{V_x}{2} + w\right] = 0$$

n.b.: B drops out as imposed $\underline{B} = B\hat{x}$

$\underline{B}_1 = \underline{B}_2$

N.b.: If oblique $\Rightarrow$ $\left.\begin{array}{l}\text{switch on}\\ \text{switch off}\end{array}\right\}$ phenomena



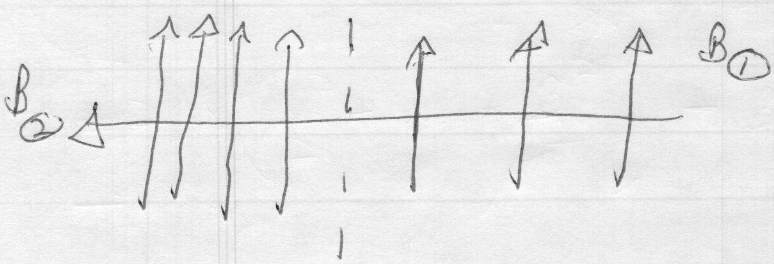
so

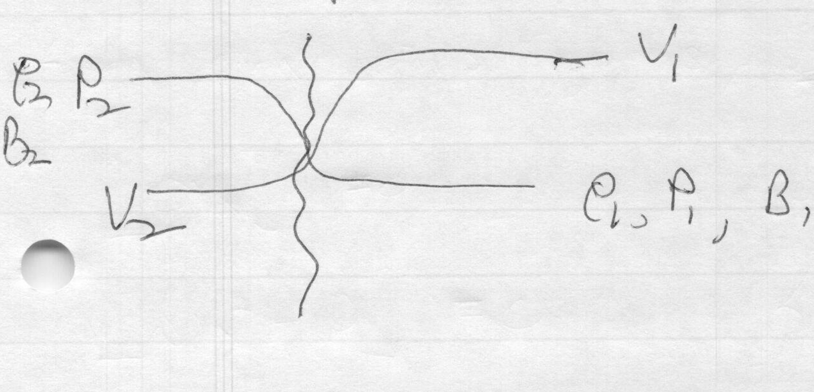
- and, as before, RH $\Rightarrow$ same results

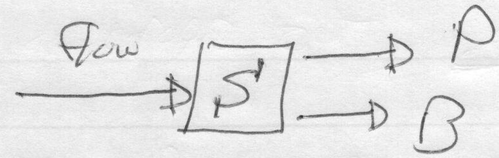
- akin to parallel propagation limit of slow wave as un-affected by B_0, in wave theory

<u>i.e.</u> $\omega^2 = k_x^2 c_s^2$

b) Perpendicular Shock $\quad\theta = \pi/2$







Some general comments:

- issue: what defines "shock", here?
(i.e. B large?)

$$V_1 > C_{S,1} \rightarrow V > V_{MS,1} \qquad V_{MS} = \left(C_S^2 + V_A^2\right)^{1/2}$$

- how much field compression possible?

(i.e. field amplification in SNR?)

point: $\rho_2/\rho_1 \leq \frac{\gamma+1}{\gamma-1}$!

so, since $B/\rho = \text{const.} \Rightarrow B_1/\rho_1 = B_2/\rho_2$

$$\therefore \quad \frac{B_2}{B_1} < \frac{\gamma+1}{\gamma-1}$$

- how much restriction on heating does $\underline{B}$ place?

$\Rightarrow$ not much, i.e. for $V/V_{MS} \equiv M_{eff} \gg 1$

should get $\simeq$ same as before

Point: $\Delta\left(B^2\right)$ constrained by compression ratio
<u>via</u> freezing in.

Proceeding:

jump conditions $\Rightarrow$

$[\rho V_n] = 0$ $\quad\quad$ $[B_n] = 0$

$\left[\rho V_n^2 + P + \frac{B^2}{8\pi}\right] = 0$ $\quad\quad$ $(\underline{B}\cdot\hat{n}) = 0$

$\left[\rho V_n\left(\frac{V^2}{2} + w + \frac{B^2}{4\pi}\right) - (\underline{B}\cdot\hat{n})(\underline{B}\cdot\underline{V})\right] = 0$

$[\hat{n} \times \underline{V} \times \underline{B}] = 0$

$\Rightarrow$ simplifies to:

$\left[\frac{V_x^2}{2} + w + \frac{B^2}{4\pi}\right] = 0$ $\quad\quad$ $[\hat{n} \times \underline{V} \times \underline{B}] = 0$

$\left[\rho V_x^2 + P + \frac{B^2}{8\pi}\right] = 0$

$[\rho V_x] = 0$ $\quad\quad$ (two non-trivial conditions)

convenient to work with: $M = V_1/C_s$
inflow Mach #

$r = \rho_2/\rho_1 = B_2/B_1 \rightarrow$ compression ratio

$R = P_2/P_1 \longrightarrow$ strength parameter

$M \rightarrow$ control parameter

$r, R \rightarrow$ output

now, for stress balance jump condition:

$$\rho_1 V_1^2 + P_1 + \frac{B_1^2}{8\pi} = \rho_2 V_2^2 + P_2 + \frac{B_2^2}{8\pi}$$

(1) (2) (3) (4) (5) (6)

(1) $\rho_1 V_1^2 = \rho_1 C_{s_1}^2 M^2$

(2) $P_1 = \frac{\rho_1 C_{s_1}^2}{\gamma}$

(3) $\frac{B_1^2}{8\pi} = \frac{\rho_1 C_{s_1}^2}{\gamma \beta}$ $\beta \equiv P_{Th}/P_M$

$\Rightarrow$

$$\rho_1 C_{s_1}^2 \left[M^2 + \frac{1}{\gamma} + \frac{1}{\gamma\beta} \right] = \rho V_2^2 + P_2 + \frac{B_2^2}{8\pi}$$

$$(4) = \frac{\rho_2 V_2^2}{\rho_1 C_{s_1}^2} = \frac{\rho_2^2 V_2^2}{\rho_2 \rho_1 C_{s_1}^2} = \frac{\rho_1^2 V_1^2}{\rho_2 \rho_1 C_{s_1}^2} = \frac{M^2}{r}$$

$$⑤ = \frac{\rho_2 c_{s_2}^2}{\gamma \rho_1 c_{s_1}^2} = \frac{R}{\gamma}$$

$$⑥ = B_2^2 / 8\pi \rho_1 c_{s_1}^2 = \frac{B_1^2 (\rho_2/\rho_1)^2}{8\pi \rho_1 c_{s_1}^2}$$ freezing in!

$$= \frac{B_1^2 r^2}{8\pi \rho_1 c_{s_1}^2}$$

$$= r^2 / \gamma \beta$$

$\Rightarrow$

$$\gamma M^2 + \frac{1}{\gamma} + \frac{1}{\gamma\beta} = \frac{M^2}{r} + \frac{R}{\gamma} + \frac{r^2}{\gamma\beta}$$

$$\Rightarrow \boxed{\gamma M^2 (1 - 1/r) = (R-1) + \frac{1}{\beta}(r^2-1)}$$

stress balance

and similarly, energy jump condition $\Rightarrow$

$$\boxed{\gamma M^2 \left(1 - \frac{1}{r^2}\right) = \frac{2\gamma}{\gamma-1}\left(\frac{R}{r} - 1\right) + \frac{4(r-1)}{\beta}}$$

proceeding:

→ eliminate R, using stress balance

$$R = 1 + \gamma M^2 (1 - 1/r) - \frac{1}{\beta}(r^2 - 1)$$

→ ~~exclude~~ plug into energy balance

→ exclude trivial root $r=1$, $R=1$ (no shock)

i.e. cancels factor $(r-1)$, $(R-1)$

∴ have for r:

$$\boxed{2(2-\gamma)r^2 + \left[2\gamma(1+\beta) + \beta\gamma(\gamma-1)M^2\right] r - \beta\gamma(\gamma+1)M^2 = 0}$$

now → 2 roots r_1, r_2

$$r_1 r_2 = -\beta\gamma(\gamma+1)M^2 / 2(2-\gamma)$$

(i.e. $(r-r_1)(r-r_2) = r^2 - (r_1+r_2)r + r_1 r_2$)

as $\gamma < 2$ $\quad r_1 r_2 < 0$

⇒ 1 value: $r_1 < 0$
1 value: $r_2 > 0$

then condition $r_2 > 1 \Rightarrow$

$$\boxed{\gamma M^2 > \gamma + 2/\beta}$$

$$\Rightarrow \gamma \frac{V_1^2}{C_{s_1}^2} > \gamma + \frac{2\,(B_1^2/8\pi)}{p}$$

$$\Rightarrow \boxed{V_1^2 > C_s^2 + V_A^2} \quad \checkmark$$

$\rightarrow$ for $\perp$ shock, inflow must beat magnetosonic speed!

For shock strength:

$$\boxed{R = 1 + \gamma M^2 (1 - 1/r) - (r^2 - 1)/\beta}$$

Note:

$\rightarrow$ for $\beta \ll 1$ (strong field)

need $M^2 \gg 1/\beta \Rightarrow V_1^2 \gg V_A^2$ for significant heating to occur.

i.e. $\perp$ magnetic field always absorbs some inflow kinetic energy.

⇒ in strong field, should use $M = V/V_{MS} \sim V/V_A$

(Alfvenic Mach #)

→ if $M_A \gg 1$,

$$R = 1 + \gamma M^2 \left(\frac{r-1}{r}\right) - (r^2-1)/\beta$$

$$\simeq \gamma M^2 \left(\frac{r-1}{r}\right)$$

and can recall from before:

$$R = \frac{P_2}{P_1} \to \frac{\gamma+1}{\gamma-1} \frac{T_2}{T_1}$$

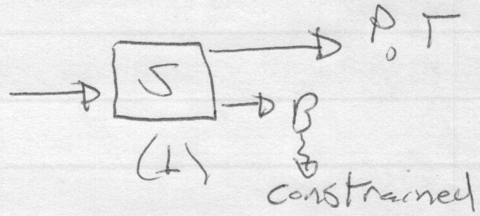
$$\therefore \frac{T_2}{T_1} \cong \frac{\gamma-1}{\gamma+1} \gamma M^2 \left(\frac{\frac{\gamma+1}{\gamma-1} - 1}{\frac{\gamma+1}{\gamma-1}}\right)$$

$$= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M^2$$

agrees with unmagnetized shock.

→ Key Message: - Freezing in ties B-compression to density compression

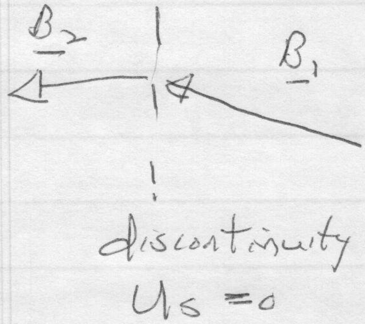
∴ - limits amount of inflow KE deposited in magnetic field i.e.



c) Oblique Shocks

"Oblique" → $\underline{V}$ inclined relative to $\underline{B}$ (not $\hat{n}$!)

i.e



(planarity → $B \times S$)

i.e. $\underline{V}_{1,2} = V_{1,2\,x}\,\hat{x}$

but $\underline{B}_{1,2} \equiv (B_x, B_{y\,1,2})$

$([B_x] = 0)$

→ point/novelty : ① jumps in B_y $\quad B_{y_1} \to B_{y_2}$

② correspondance with oblique fast, slow wave (→ fast, slow shock ?!)

fast (⊥ → magn.-sonic)
slow (∥ → acoustic)

[n.b. intermediate wave → rotational discontinuity]

→ proceed : take $V_n = V_x$
and grind along, with B_x, $B_{y_{1,2}}$

→ or use a trick ?

trick : use of Teller / De Hoffmann frame

introduce a V_{y_1} ! $\left\{ \begin{array}{l} \text{gain } V_{y_1}, \text{ but} \\ \text{eliminat } \underline{E} \times \underline{B} \end{array} \right.$

- here, convenient to introduce (as can bound r) frame of motion in V_{y_1}

i.e. choose frame with V_{y_1} s/t

$$\underline{E}_1 = -\frac{\underline{V}_1 \times \underline{B}}{c} = 0$$

(surface of discontinuity still at $x=0$)

$\Rightarrow$

$$V_{y_1} = \frac{V_{x_1} B_{y_1}}{B_{x_1}} = \frac{V_{x_1} B_{y_1}}{B_x}$$ indiv

as $[B_x] = 0$

- $\underline{V}_1$ s/t $\underline{E}_1 = 0 \Rightarrow \underline{V}_{y_1} = V_{y_1} \hat{y}$

$\Rightarrow$ (F. De Hoffmann, E. Teller, 1950)

- trade-off ? - gain $V_{y_{1,2}}$ and concommitant $[\rho V_x V_y]$ (would have anyway, due $\underline{J} \times \underline{B}$ impulse!)

frame change only a trade-off.

- lose $\frac{c}{4\pi}(\underline{E} \times \underline{B})$ in energy jump,

since $\underline{E}_1 = 0$ and

$[\underline{V} \times \underline{B}] = 0$

- so, $V_{y_1} \equiv V_{x_1} B_y / B_x$

$[E_t] = 0$, as $[\hat{n} \times \underline{v} \times \underline{B}] = 0$

$[(\underline{V} \times \underline{B})_t] = 0$

$\Rightarrow \quad \underline{E}_{t②} = 0$

$$\Rightarrow \quad V_{y_2} = \frac{V_{x_2} B_{y_2}}{B_x}$$

so

$$\frac{V_{y_2}}{V_{y_1}} = \frac{V_{x_2} B_{y_2}}{\cancel{B_x} \frac{V_{x_1} \cancel{B_{y_1}}}{\cancel{B_x}}}$$

$$= \frac{V_{x_2}}{V_{x_1}} \frac{B_{y_2}}{B_{y_1}} = \frac{\rho_1}{\rho_2} \frac{B_{y_2}}{B_{y_1}} \qquad [\rho v_n] = 0$$

$$\boxed{\frac{V_{y_2}}{V_y} = \frac{1}{r} \frac{B_{y,2}}{B_{y,1}}}$$

$[E_t] = 0 \Rightarrow B_x V_{y_1} - V_{x_1} B_{y_1} = B_x V_{y_2} - V_{x_2} B_{y_2}$

$$\Rightarrow \boxed{V_{y_1} - \frac{V_{x_1} B_{y_1}}{B_x} = V_{y_2} - \frac{V_{x_2} B_{y_2}}{B_x}}$$

(redundant with above)

(hold)

now, as oblique, also have:

$$\left[\rho V_x V_y - \frac{B_x B_y}{4\pi}\right] = 0$$

$$\Rightarrow \quad \rho_2 V_{x_2} V_{y_2} - \rho_1 V_{x_1} V_{y_1} = \frac{B_{x_2} B_{y_2}}{4\pi} - \frac{B_{x_1} B_{y_1}}{4\pi}$$

div thru

$$\overbrace{\frac{\rho_2 V_{x_2}}{\rho_1 V_{x_1}}}^{1} \frac{V_{y_2}}{V_{y_1}} - 1 = \left(\frac{1}{\rho V_{x_1} V_{y_1}}\right)\left(\frac{B_{x_2} B_{y_2}}{4\pi} - \frac{B_{x_1} B_{y_1}}{4\pi}\right) \quad \cancel{\rho_1 V_{x_1} V_{y_1}}$$

$$\frac{V_{y_2}}{V_{y_1}} - 1 = \left(\frac{B_x B_{y_1}}{4\pi \rho_1 V_{x_1} V_{y_1}}\right)\left(\frac{B_{y_2}}{B_{y_1}} - 1\right)$$

$V_{y_1} = V_{x_1} B_{y_1} / B_{x_1} \quad \Rightarrow$

$$\boxed{\frac{V_{y_2}}{V_{y_1}} - 1 = \frac{B_{x_1}^2}{4\pi \rho_1 V_{x_1}^2}\left(\frac{B_{y_2}}{B_{y_1}} - 1\right)}$$

so

$$\boxed{\begin{aligned} \frac{V_{y_2}}{V_{y_1}} - 1 &= \frac{B_{x_1}^2}{4\pi\rho_1 V_{x_1}^2}\left(\frac{B_{y_2}}{B_{y_1}} - 1\right) \\ \frac{V_{y_2}}{V_{y_1}} &= \frac{V_{x_2}}{V_{x_1}}\frac{B_{y_2}}{B_{y_1}} = \frac{1}{r}\frac{B_{y_2}}{B_{y_1}} \end{aligned}}$$

→ t_{xy} Π.

→ frame

$$V_A^2 \equiv B_{x_1}^2/4\pi\rho_1$$

⇒

$$\frac{V_{y_2}}{V_{y_1}} = 1 + \frac{V_A^2}{V_{x_1}^2}\left(\frac{B_{y_2}}{B_{y_1}} - 1\right)$$

$$= 1 + \frac{V_A^2}{V_{x_1}^2}\left(r\frac{V_{y_2}}{V_{y_1}} - 1\right)$$

⇔

$$\boxed{V_{y_2}/V_{y_1} = \frac{V_{x_1}^2 - V_A^2}{V_{x_1}^2 - rV_A^2} = \frac{1}{r}\frac{B_{y_2}}{B_{y_1}}}$$

Now, finally use $\left[w + \frac{V^2}{2}\right] = 0$ $\quad(\underline{E} = 0)$

$V^2 = V_x^2 + V_y^2$ (both components!)

note: r undetermined here ⟹ normal stress

so plugging in $(w = \gamma P/\rho(\gamma-1))$

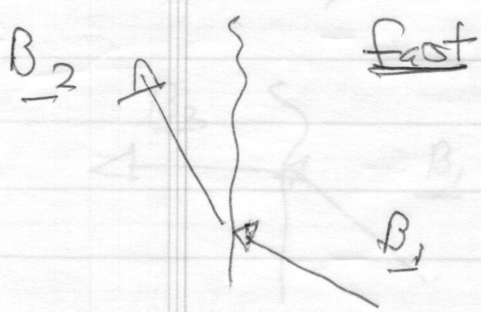
⟹

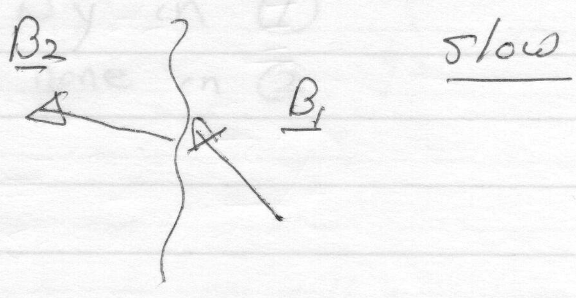
$$\boxed{\begin{aligned} \frac{P_2}{P_1} &= r + (\gamma-1)\frac{r V_{x_1}^2}{2c_{s_1}^2}\left[1 - \frac{\cos^2\theta}{r} - \sin^2\theta\left(\frac{V_{x_1}^2 - V_A^2}{V_{x_1}^2 - rV_A^2}\right)\right] \\ V_{y_2}/V_{y_1} &= \frac{V_{x_1}^2 - V_A^2}{V_{x_1}^2 - rV_A^2} = \frac{1}{r}\frac{B_{y_2}}{B_{y_1}} \end{aligned}}$$

so, the answer (obliquity phenomena?):

$B_{y_2} > B_{y_1}$ if $V_{x_1}^2 > rV_A^2 > V_A^2$ ⟹ "fast shock"

$B_{y_2} < B_{y_1}$ if $V_{x_1}^2 \lesssim V_A^2 < rV_A^2$ ⟹ "slow shock"





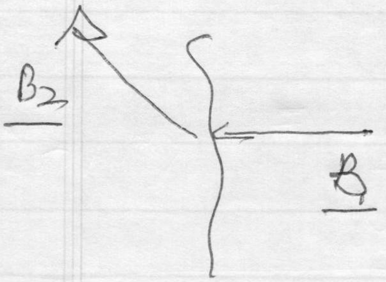
refract away normal refract toward normal

what of equalities?

refraction ⟺ critical speed

if $V_{x_1}^2 = r V_A^2$

$B_{1y} = 0$
$B_{2y} \neq 0$ $\Rightarrow$ "switch on" shock

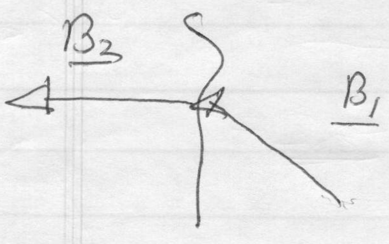


no B_y in 1
$B_y \neq 0$ in 2

if $V_{x_1}^2 = V_A^2$

$B_{y_1} \neq 0$
$B_{y_2} = 0$ $\Rightarrow$ "switch off" shock

B_y in 1
none in 2



Note: switch-off $\Leftrightarrow$

- for $\theta = 0$ $(B_{1y} = 0)$ $V_1 = V_A$, $B_{2y} = 0$

so $\underline{B}_1 \parallel \underline{B}_2 \parallel \hat{n}$

$\boxed{[B_n]} = 0 \Rightarrow \underline{B}_1 = \underline{B}_2 \rightarrow$ reduces to parallel and hydro case!

⇒ What of Intermediate Wave ?

⇒ non-shock discontinuity ⇒ i.e. no heating, etc.

⇔ rotational discontinuity !

→ $j = 0$ (no mass flow thru discontinuity) contrast shock.

→ magnetic field rotates thru an angle in jump

⇒ better understood in context of collisionless shocks

⇒ RD -type structures are seen in solar wind, etc. (i.e. Ulysseus)

This brings us to . . . .

<u>Collisionless Shocks !</u>

Ion - Acoustic Shocks and Solitons - { simplest form c-shock

In quasi-neutral system ($k^2 \lambda_{De}^2 << 1$)

$$n_e = n_0 \exp\left[|e| \phi / T_e\right]$$

$$\frac{\partial n_i}{\partial t} + v \frac{\partial n_i}{\partial x} = -n_i \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -|e| \frac{\partial \phi}{\partial x}$$

$$\phi = \frac{T_e}{|e|} \ln\left(n_e / n_0\right) = \frac{T_e}{|e|} \ln\left(n_i / n_0\right)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} = \frac{T_e}{|e|} \frac{\cancel{n_0}}{n_i} \frac{1}{\cancel{n_0}} \frac{\partial n_i}{\partial x}$$

$\Rightarrow \quad \dfrac{\partial n_i}{\partial t} + \dfrac{\partial}{\partial x}(n_i V) = 0$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} = -\frac{T_e}{n_i}\frac{\partial n_i}{\partial x}$$

→ isomorphic to 1D gas-dynamic equations (albeit isothermal)

→ steepening, shock formation will occur

→ but, as dissipation miniscule, shock limited by dispersion, not dissipation

i.e. isomorphism to gas dynamics ⇒ $k^2\lambda_D^2 \ll 1$ (quasi-neutrality)

Shock structure limited when $L \sim \lambda_{De}$
→ Quasi-neutrality violated!

i.e. allowing for dispersion:

$n_e = \exp\left[|e|\phi/T_e\right]$ Boltzmann Electrons

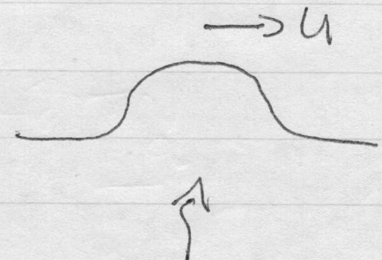
$$\left.\begin{aligned} &\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i V) = 0 \\ &\partial V_i/\partial t + V_i \partial V_i/\partial x = -\frac{|e|}{m_i}\frac{\partial \phi}{\partial x} \end{aligned}\right\} \text{fluid ions}$$

$\tilde{n}_e = \exp(q\phi/T_e)$

$\tilde{n}_i$: $\quad \frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i V) = 0$

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} = -\frac{q}{m_i}\frac{\partial \phi}{\partial x}$$

$$\left\{\begin{matrix} n_i \\ V_i \\ \phi \end{matrix}\right\} = F(x - ut)$$



i.e. localized solution, moving at u.

$$-u n_i' + (n_i V)' = 0$$

$$(V-u)V' = -\frac{q}{m_i}\phi'$$

Now, integrating with $\left.\begin{matrix} \phi \to 0 \\ V \to 0 \\ n \to n_0 \end{matrix}\right\}$ $x \to \infty$

$\Rightarrow$

$$-u n_i + n_i V = -u$$

$\Rightarrow (u - V) n_i = u \;\rightarrow$ to ensure $n \to n_0$

$$n_i = u/(u-V)$$

Similarly,

$$\frac{q\phi}{m_i} = -\frac{1}{2}(u-V)^2 + \frac{u^2}{2} \qquad \left(\text{to ensure } \phi \to 0\right)$$

$$\Rightarrow \left(\frac{1}{2}u^2 - \frac{q}{m_i}\phi\right) = \frac{1}{2}(u-V)^2$$

$$\Rightarrow (u-V) = \left(u^2 - \frac{2q\phi}{m_i}\right)^{1/2}$$

so $$\frac{\partial^2\phi}{\partial x^2} = -4\pi n_0 q \left(\frac{1}{\left(1-\frac{2q\phi}{m_i u^2}\right)^{1/2}} - \exp(q\phi/T_e)\right)$$

$$\phi'\phi'' = -4\pi n_0 q\, \phi' \left(\frac{1}{\left(1-\frac{2q\phi}{m_i u^2}\right)^{1/2}} - \exp\left(\frac{q\phi}{T_e}\right)\right)$$

integrating $\Rightarrow$

$$\frac{1}{2}\phi'^2 + V(\phi) = 0$$

$$\boxed{V(\phi) = -4\pi n_0 \left\{ mu\left(u^2 - \frac{2q\phi}{m}\right)^{1/2} + T_e e^{q\phi/T_e}\right\} + C}$$

$\hookrightarrow$ Sagdeev Potential

$$\phi'' = dV/d\phi$$

Collisionless shock $\Leftrightarrow$ solitary wave

$\rightarrow$ Ion acoustic soliton reduced to particle orbit pblm!

Define Mach # $\mathcal{M} = u/c_s$

$$u = \mathcal{M} c_s$$

$\Rightarrow$

$$V(\phi) = -4\pi n_0 \left\{ m \mathcal{M} c_s \left(\mathcal{M}^2 c_s^2 - \frac{2q\phi}{m}\right)^{1/2} + T_e e^{q\phi/T_e} \right\} + C$$

$$= -4\pi n_0 \left\{ T_e \mathcal{M} \left(\mathcal{M}^2 - \frac{2q\phi}{T_e}\right)^{1/2} + T_e e^{q\phi/T_e} \right\} + C$$

Thus:

$\rightarrow$ need $\mathcal{M}^2 > 2q\phi_M/T_e$ for soliton to exist
(reality)

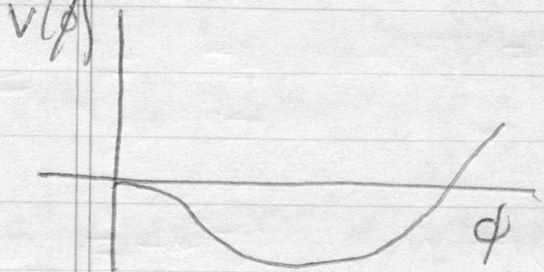
$\therefore$ $\dfrac{u^2}{c_s^2} > 2q\phi_M/T_e$ (critical velocity)
$\rightarrow$ speed – amplitude connection

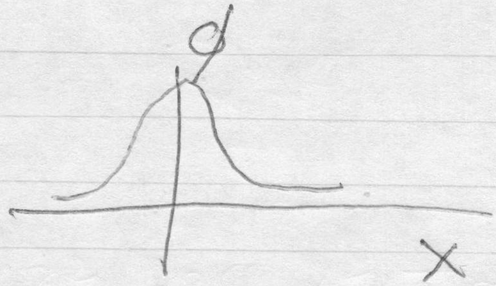
$\rightarrow$ Similarly, for small ϕ,

$$V(\phi) \cong -4\pi n_0 \left\{ T_e \mathcal{M}^2 \left(1 - \frac{q\phi}{\mathcal{M}^2 T_e} - \frac{1}{2}\left(\frac{q\phi}{T_e \mathcal{M}^2}\right)^2\right) + T_e + q\phi + \frac{T_e}{2}\left(\frac{q\phi}{T_e}\right)^2 \right\}$$

$$\cong -4\pi n_0 \left\{ T_e(1+\mathcal{M}^2) + \cancel{q\phi} - \cancel{q\phi} + \frac{T_e}{2}\left(\frac{q\phi}{T}\right)^2 \left(-\frac{1}{\mathcal{M}^2} + 1\right) \right\}$$

Now, for soliton ⇒ need bound state





Then $V''(\phi)\big|_{\phi\to 0} < 0 \quad \Rightarrow \quad \mathcal{M}^2 > 1$

So need $\mathcal{M} > 1$ for soliton formation.

→ Similarly, need $\mathcal{M} \lesssim 1.6$

∴ for soliton, need $1 < \mathcal{M} < 1.6$
($e\phi/T$ small)

i.e. have

$$V(\phi) = -4\pi n_0 \left\{ m u \left(u^2 - \frac{2e\phi}{m}\right)^{1/2} + T_e\, e^{e\phi/T} \right\}$$
$$= -\phi'^2$$

take ϕ_{max} when $V(\phi) = 0 \rightarrow \phi' \equiv 0$

⇒ defines ϕ_{max}

More Generally:

→ as dissipation miniscule, shock limited by dispersion, not dissipation

i.e. quasi-neutrality ⟺ $k^2\lambda_{De}^2 \ll 1$

When $L_{shock} \sim \lambda_{De}$ ⇒ quasineutrality violated!

∴ ion-acoustic shock limited by dispersion

i.e. $\omega^2 = k^2 c_s^2 / 1 + k^2\lambda_{De}^2$

→ Generally, sub-classify shocks into:

collisional → old standard hydrodynamics
L_{shock} limited by dissipation

collisionless → a/a' ion-acoustic in plasma
L_{shock} limited by dispersion
⇒ forms soliton

Aside: Some Generic Properties of Solitons

Contrast → Sound wave $\omega = k c_s$
$x = (c_s + V)t + f(V)$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + c_s\frac{\partial V}{\partial x} = 0$$

$\Rightarrow$ Dispersive Ion Acoustic Wave

$$\omega^2 = k^2 c_s^2 / 1 + k^2 \lambda_D^2$$

$k\lambda_D < 1 \Rightarrow \quad \omega = k c_s \left(1 - k^2 \lambda_D^2 / 2\right) \qquad \left(\omega \text{ odd in } k\right)$

suggests model equation of form:

$$\frac{\partial \mathcal{E}}{\partial t} + (c_s + \mathcal{E}) \frac{\partial \mathcal{E}}{\partial x} + \frac{c_s \lambda_D^2}{2} \frac{\partial^3 \mathcal{E}}{\partial x^3} = 0$$

of generic form:

$$\frac{\partial \mathcal{E}}{\partial t} + u_0 \frac{\partial \mathcal{E}}{\partial x} + \alpha\, \mathcal{E} \frac{\partial \mathcal{E}}{\partial x} + \beta \frac{\partial^3 \mathcal{E}}{\partial x^3} = 0$$

$$a \equiv \alpha \mathcal{E}$$
$$y = x - u_0 t$$

$$\Rightarrow \boxed{\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial y} + \beta \frac{\partial^3 a}{\partial y^3} = 0} \qquad \left(\text{Korteweg-deVries Eqn. (KdV)}\right)$$

dispersion

vs

dissipation

contrast

$$\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial y} - \bar{\nu} \frac{\partial^2 a}{\partial y^2} = 0 \qquad \left(\text{Burgers Eqn.}\right)$$

Burgers → dissipative ($\bar{\nu}$ limits steepening)

$$L_{shock} \sim \bar{\nu}/a$$

KdV → dispersive (ω variation with k ⇒ u variation with k limits steepening – diff'nt scale comp.)

$$L_{soliton} \sim (\beta/a)^{1/2}$$

Solution of KdV Equation:

$$\frac{\partial a}{\partial t} + a\frac{\partial a}{\partial y} + \beta\frac{\partial^3 a}{\partial y^3} = 0$$

$$a = a(y - V_0 t) \qquad \Rightarrow \quad V_{wave} = u_0 + V_0$$

⇒

$$\beta a''' + a a' - V_0 a' = 0$$

$\left\{ \begin{array}{l} \text{Invariant} \\ a \to a + V \\ V_0 \to V_0 + V \end{array} \right.$

∴

$$\beta a'' + \frac{1}{2}a^2 - V_0 a = \frac{1}{2} C_1$$

$$2\beta a'' a' + a' a^2 - 2V_0 a' a = C_1 a' \qquad (* \, 2a')$$

$$\Rightarrow \quad \beta a'^2 = -\frac{1}{3}a^3 + V_0 a^2 + C_1 a + C_2$$

+ can reduce to quadrature

Convenient to factorize:

$V_0, c_1, c_2 \rightarrow a_1, a_2, a_3$

$$\Rightarrow \beta a'^2 = -\frac{1}{3}(a-a_1)(a-a_2)(a-a_3)$$

where $V_0 = \frac{1}{3}(a_1 + a_2 + a_3)$

For $\rightarrow$ bounded $|a(y)|$

$\rightarrow$ need a_1, a_2, a_3 real

if $a_1 > a_2 > a_3$

$$\Rightarrow a_1 \geq a \geq a_2 \qquad (\beta a'^2 > 0)$$

$\therefore$ $a_3 = 0$ is no loss generality

$$\Rightarrow \beta a'^2 = \frac{1}{3}(a_1 - a)(a - a_2)\, a$$

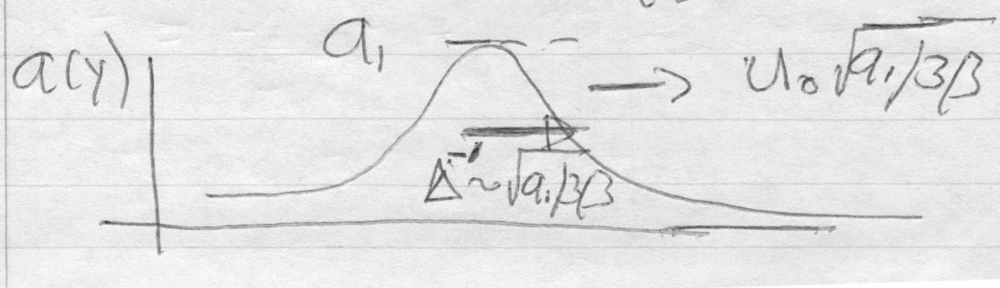
if $a_2 = 0$

Exact solution of NL KdV Eqn.

$$\therefore \boxed{a(y) = a_1 \cosh^{-2}\left(\frac{1}{2} y \sqrt{a_1/3\beta}\right)}$$

$$= a_1 \cosh^{-2}\left(\frac{1}{2}(x - u_0 t)\sqrt{a_1/3\beta}\right)$$

$\Rightarrow$



→ soliton has finite width

$\Delta \sim \sqrt{3\beta / a_1}$ $\qquad \beta \sim \lambda_{De}^2$ for IA.

$\Rightarrow \Delta \sim \lambda_D$

⟷ contrast zero-width shock

→ soliton has finite amplitude a_1

with $V \sim u_0 \sqrt{a_1/3\beta}$

∴ bigger solitons move faster!

Note: $a_2 \neq 0 \Rightarrow$ non-localized, oscillatory solution.

General comments:

→ Collisional shock $\Delta \sim \nu / a$

Collisionless Shock $\Delta \sim \lambda_D \sqrt{c_s/a}$

∴ Debye length sets discontinuity scale

→ Can treat collisionless shock via

$\nabla^2 \phi = -4\pi n_0 q (\tilde{n}_i - \tilde{n}_e)$

etc $\Rightarrow$ Sagdeev Potential